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ON THE PROPERTIES OF THE CONTROLLABILITY SET FOR DIFFERENTIAL INCLUSION UNDER CONDITION MOBILITY OF TERMINAL SET

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Abstract: In this paper we consider the model of dynamic system in the form a differential inclusion. The property of controllability of this system under conditions mobility of terminal set M is researched. For one class differential inclusions the structural properties of the set of M-controllability are studied.

Keywords: differential inclusion, control system, terminal set, controllability, structural properties.

1. Introduction.

Differential equations with a multi-valued right-hand side are differential inclusions, i.e. relations of the form

$$\frac{dx}{dt} \in F(t,x), \qquad (1)$$

where $x = x(t)$ – the desired n-vector function, is of great interest as mathematical models of various dynamical systems. They arise in control theory, in the theory of differential equations with discontinuous right-hand sides, in differential games, in mathematical economics, and in other areas of applied mathematics.

The theory of differential inclusions, which is a modern branch of mathematics, develops in various directions and has numerous applications. A large class of differential inclusions is controlled differential inclusions [4-6], which are of important interest in control problems under conditions of information inaccuracy and uncertainty of parameters of various types. Methods of the theory of differential inclusions are developed in close connection with the theory of multi-valued maps, convex and nonsmooth analysis [1-3].

Differential inclusions are a convenient and effective mathematical tool for studying many important issues of control theory, such as the structural properties of the reachability set and its continuous dependence on parameters, the existence of optimal control, necessary and sufficient conditions of optimality [1,7], etc.

2. Problem statement. Research methods.

For dynamical systems, the question of controllability is of particular interest, i.e., the property of the system, which is expressed by the possibility of reaching the terminal state with the help of controlled movements – trajectories emerging from a set of initial states. It is convenient to study this question with the help of a model of a control system in the form of a differential inclusion (1). Therefore, for dynamical systems, one of the topical issues is the property of controllability of the trajectories of differential inclusions [7].

By the trajectories of the differential inclusion (1) we mean every absolutely continuous *n*-vector function $x = x(t)$, $t \in T = [t_0, t_1]$, satisfying almost everywhere on $T = [t_0, t_1]$ the relation $\frac{dx(t)}{dt} \in F(t, x(t))$.

The differential inclusion (1) is called controllable from the initial state x_0 to the final state x_1 ("pointwise" controllable) if there exists a trajectory $x(t)$ defined on some segment $T = [t_0, t_1]$, such that $x(t_0) = x_0$, $x(t_1) = x_1$.

Definition 1. The sets of zero-controllability of the differential inclusion are called the set of all those points $x_0 \in R^n$ from which the origin of coordinates is achievable along the trajectories ($x(t_0) = x_0$, $x(t_1) = 0$) of the differential inclusion (1).

Let be a mobile, i.e. time-dependent terminal set $M = M(t), t \geq t_0$. By analogy with the concept of a zero-controllability set, we can introduce the concept of $M-$controllability for the case of a mobile terminal set as follows.

Definition 2. Point $x_0 \in R^n, x_0 \notin M(t_0)$, we call the point of $M-$controllability of the differential inclusion (1) for a given mobile terminal set $M = M(t)$, if there exist a absolutely continuous trajectory $x(t)$ defined on some segment $T = [t_0, t_1]$ such that $x(t_0) = x_0$, $x(t_1) \in M(t_1)$.

We denote by $W(M, F)$ the set of all points of $M-$controllability of the differential inclusion (1). The main goal of this paper is to study such properties of the set $W(M, F)$ that clarify its topological structure.

Let $X_T(t_0, t_1, x_0, F)$ be the set of reachability of the differential inclusion (1) from the starting point $x_0 \in R^n$ at time $t_1 > t_0$, i.e. the set of possible points $x_1 \in R^n$ for which there exist trajectories $x = x(t), t \in T = [t_0, t_1]$, such that $x(t_0) = x_0$ and $x(t_1) = x_1$. From definition 2, it is clear that point $x_0 \in R^n$ is the point of $M-$controllability of the differential inclusion (1) if and only if there exists $t_1 > t_0$ such that $X_T(t_0, t_1, x_0, F) \cap M(t_1) \neq \emptyset$, where $T = [t_0, t_1]$, $x_0 \notin M(t_0)$.

So, it is clear that, in order to study the properties of the controllability set of the differential inclusion (1), it is necessary to study the structure of the set

$$K(t_0, t_1, M, F) = \left\{\xi \in R^n : X_T(t_0, t_1, \xi, F) \cap M(t_1) \neq \emptyset\right\}$$

at $t_1 > t_0$, taking into account properties $M = M(t)$ and $F = F(t, x)$.

From the definition of sets $W(M, F)$ and $K(t_0, t_1, M, F)$, the validity of the following equality easily follows

$$W(M, F) = (\bigcup_{t_1 > t_0} K(t_0, t_1, M, F)) \setminus M(t_0). \qquad (2)$$

Obviously, if $F_1(t, x) \subset F_2(t, x)$, then $X_T(t_0, t_1, \xi, F_1) \subset X_T(t_0, t_1, \xi, F_2)$ and at $M_1(t) \subset M_2(t), t \geq t_0$, from the relation $X_T(t_0, t_1, \xi, F_1) \cap M_1(t_1) \neq \emptyset$ follows $X_T(t_0, t_1, \xi, F_2) \cap M_2(t_1) \neq \emptyset$. Therefore,

$$K(t_0, t_1, M_1, F_1) \subset K(t_0, t_1, M_2, F_2), \quad W(M_1, F_1) \subset W(M_2, F_2).$$

Hence, in particular, we get that if there are maps $A: R^1 \to R^{nxn}$, $B: R^1 \to \Omega(R^n)$, such that $A(t)x + B(t) \subset F(t, x)\ \forall (t, x) \in R^1 \times R^n$, then to check the property of $M-$controllability of the differential inclusion (1), it is sufficient to check the M –controllability of the differential inclusion

$$\dot{x} \in A(t)x + B(t). \tag{3}$$

3. Main results.

Let us study the structural properties of the set of M –controllability of the differential inclusion (3). According to the accepted notation $W(M, A, B)$ there is a set of all points of M – controllability of the differential inclusion (3) for a given terminal set $M = M(t), t \geq t_0$. Further, denoting $X_T(t_0, t_1, \xi, A, B)$ the set of reachability of the differential inclusion (3), $K(t_0, t_1, M, A, B)$ the set is defined similarly to the set $K(t_0, t_1, M, F)$, i.e.

$$K(t_0, t_1, M, A, B) = \{\xi \in R^n : X_T(t_0, t_1, \xi, A, B) \cap M(t_1) \neq \varnothing\}.$$

Since, according to (2)

$$W(M, A, B) = (\bigcup_{t_1 > t_0} K(t_0, t_1, M, A, B)) \setminus M(t_0),$$

then the structural properties of set $W(M, A, B)$ are expressed in terms of similar properties of sets of the form $K(t_0, t_1, M, A, B)$.

In the future, we will assume that the elements of the matrix $A(t)$ are measurable on any $T = [t_0, t_1] \subset [t_0, +\infty]$ and $\| A(t) \| \leq a(t)$, where $a(\cdot) \in L_1(T)$, and the multi-valued map $t \to B(t) \in \Omega(R^n)$ is measurable on any segment $T = [t_0, t_1] \subset [t_0, +\infty]$ and $\| B(t) \| \leq b(t)$, where $b(\cdot) \in L_1(T)$.

It is well known [8] that for every integrable function $b: T \to R^n$, the absolutely continuous solution of equation $\dot{x} = A(t)x + b(t), t \in T, x(t_0) = \xi$ is represented by the Cauchy formulas

$$x(t) = \Phi_A(t, t_0)\xi + \int_{t_0}^{t} \Phi_A(t, \tau) b(\tau) d\tau, t \in T. \tag{4}$$

where $\Phi_A(t, \tau)$ is the fundamental matrix of solutions to equation $\dot{x} = A(t)x, t \in T$.

The relation of $X_T(t_0, t_1, \xi, A, B) \cap M(t_1) \neq \varnothing$ is equal to the inclusion of $0 \in X_T(t_0, t_1, \xi, A, B) - M(t_1)$. Therefore,

$$K(t_0, t_1, \xi, A, B) = \{\xi \in R^n : 0 \in X_T(t_0, t_1, \xi, A, B) - M(t_1)\}.$$

Now, using the last equality and formula (4), we can get the following result.

Theorem 1. The set $K(t_0, t_1, M, A, B)$ is represented by the formula

$$K(t_0, t_1, M, A, B) = -\int_{t_0}^{t_1} \Phi_A(t_0, t) B(t) dt + \Phi_A(t_0, t_1) M(t_1) \tag{5}$$

Corollary 1. If $1\, M(t_1)$ is a convex compact, then $K(t_0, t_1, M, A, B)$ is also a convex compact of R^n . If $M(t_1)$ and $conv B(t)$ are strictly convex at $t \in T = [t_0, t_1]$, then $K(t_0, t_1, M, A, B)$ is strictly convex.

Let's say: $K_0(t_1, A, B) = K(t_0, t_1, \{0\}, A, B)$. Then it is clear from formula (5) that

$$K_0(t_0, t_1, A, B) = -\int_{t_0}^{t_1} \Phi_A(t_0, t) B(t) dt. \tag{6}$$

The set $K_0(t_0,t_1,A,B)$ is a convex compact of R^n. Taking into account the equality (6), the formula (5) takes the form:

$$K(t_0,t_1,M,A,B)=K_0(t_0,t_1,A,B)+\Phi_A(t_0,t_1)M(t_1). \qquad (7)$$

If $M(t_1)$ is a convex compact, then equality (7) can be written as the geometric difference

$$K(t_0,t_1,M,A,B)\overset{*}{-}K_0(t_0,t_1,A,B)=\Phi_A(t_0,t_1)M(t_1).$$

Let $X_T^0(t_0,t_1,A,B)=X_T(t_0,t_1,0,A,B)$ be the reachability set of system (3) at $x_0=0$.

Corollary 2. The formula is valid

$$K(t_0,t_1,M,A,B)=-\Phi_A(t_0,t_1)[X_T^0(t_0,t_1,A,B)+M(t_1)].$$

Theorem 2. Let $t_0<\bar{t}<t_1$. Then:

$$K(t_0,t_1,M,A,B)=K(t_0,\bar{t},\overline{M},A,B),$$

where $\overline{M}(t)=K(t,t_1,M,A,B)$.

In fact, using the formula (5), we have:

$$K(t_0,t_1,M,A,B)=-\int_{t_0}^{t_1}\Phi_A(t_0,t)B(t)dt+\Phi_A(t_0,t_1)M(t_1)=-\int_{t_0}^{\bar{t}}\Phi_A(t_0,t)B(t)dt+$$

$$+\Phi_A(t_0,\bar{t})[-\int_{\bar{t}}^{t_1}\Phi_A(\bar{t},t)B(t)dt+\Phi_A(\bar{t},t_1)M^0]=-\int_{t_0}^{\bar{t}}\Phi_A(t_0,t)B(t)dt+\Phi_A(t_0,\bar{t})K(\bar{t},t_1,M,A,B).$$

Corollary 3. Let $t_0<\bar{t}<t_1$. Then the relation

$$K(t_0,t_1,M,A,B)\subset K(t_0,\bar{t},M,A,B)$$

holds if and only if $K(\bar{t},t_1,M,A,B)\subset M(\bar{t})$.

Theorem 3. Let $A(t)\equiv A, B(t)\equiv B$ at $t\in T=[t_0,t_1]$. Then:

$$K(t_0,t_1,M,A,B)=\bigcup_{\xi\in M(t_1)}X(t_0,t_1,\xi,-A,-B). \qquad (8)$$

In fact, using the Cauchy formula (4), we can write the following representation

$$\bigcup_{\xi\in M(t_1)}X(t_0,t_1,\xi,-A,-B)=\Phi_{-A}(t_1,t_0)M(t_1)-\int_{t_0}^{t_1}\Phi_{-A}(t_1,t)Bdt.$$

It is not difficult to see that $\Phi_{-A}(t_1,t_0)=\Phi_A(t_0,t_1)$, $\Phi_{-A}(t_1,t)=\Phi_A(t,t_1)$, $\Phi_A(t_1+t_0-s,t_1)=\Phi_A(t_0,s)$ at $s\in[t_0,t_1]$. Now, taking into account these relations and making the substitution of variables $s=t_1+t_0-t$ in the integral

$$\int_{t_0}^{t_1}\Phi_{-A}(t_1,t)Bdt,$$

we get:

$$\bigcup_{\xi\in M(t_1)}X(t_0,t_1,\xi,-A,-B)=\Phi_A(t_0,t_1)M(t_1)-\int_{t_0}^{t_1}\Phi_A(t_0,s)Bds.$$

By virtue of theorem 1, the right-hand side of the last equality is the set $K(t_0,t_1,M,A,B)$.

4. Discussion of the results and conclusion.

The paper focuses on studying the properties of the auxiliary set $K(t_0,t_1,M,F)$, which can be used to study the properties of the M-controllability set $W(M,F)=(\bigcup_{t_1>t_0}K(t_0,t_1,M,F))\setminus M(t_0)$. From the results obtained, we should note theorem 1, which gives the formula (5) for the representation of the set $K(t_0,t_1,M,F)$ for the differential inclusion (3). This result allows us to find out some properties of the set of M-controllability. In particular, the conditions of convexity and compactness of the set $K(t_0,t_1,M,A,B)$ are specified. Theorem 2 and its corollary give an idea of the dynamics of the set $K(t_0,t_1,M,A,B)$. In Theorem 3, the formula (8) is given, which indicates a close connection of the sets of *M*-controllability with the set of reachability of the differential inclusion (3) at $A(t) \equiv A$, $B(t) \equiv B$.

In this paper, the problem of controllability of the trajectories of differential inclusions for the case of mobility of the terminal set *M*. The studied properties of the set $K(t_0,t_1,M,F)$ allow us to clarify the structure of the set *M*-controllability of the considered class of differential inclusions. The obtained results are developed by the research work [9].

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