ON THE STRUCTURAL PROPERTIES OF THE REACHABILITY SET OF A DIFFERENTIAL INCLUSION

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Abstract:

In the paper considers a mathematical model of the control object–differential inclusion. For one class of differential inclusions, the properties of a set of absolutely continuous solutions are studied. The structure of the reachability set and the reachability sphere are investigated. The conditions of convexity and compactness of these sets are specified.

Keywords:

Differential inclusion, multi-valued mapping, reachability set, reachability sphere, convexity property, compactness property.

1. Introduction.

The actuality of management and optimization problems arising in various spheres of economy, production, engineering and ecology determines the increasing importance of the mathematical theory of optimal processes [1-3]. In modern mathematical research on controlled processes, the theory of differential inclusions occupies an important place and has numerous applications [3-9].

Let R^n be an Euclidean space of n-dimensional vectors $x=(x_1,...,x_n)$, $F: R^1\times R^n \to P(R^n)$ be a given multi-valued map. We consider the Cauchy problem for differential inclusion:

$$\dot{x}\in F(t,x)\,, \quad x(t_0)=x^0\,. \qquad (1)$$

The solution of the differential inclusion (1) is understand as an absolutely continuous n-vector function $x=x(t)$ on a certain segment $T=[t_0,t_1]$, which satisfies the relation $\dot{x}(t)\in F(t,x(t))$ almost every on T.

The existence of a solution to problem (1) has been studied by many researchers. In particular, the following statement is known.

Let the conditions be met for almost all $t\in[t_0,t_0+a]$ and all $x\in S_r(x_0), x_0\in R^n$: 1) $F(t,x)\in C\Omega(R^n)$; 2) the multi-valued mapping $F(t,x)$ is upper semicontinuous by $x\in S_r(x_0)$; 3) there is a vector function $f(t,x)$ measurable in $t\in[t_0,t_0+a]$ such that $f(t,x)\in F(t,x)$; 4) there is a function $g(t)$ summable on $[t_0,t_0+a]$ such that $\|f(t,x)\|\le g(t)$. Then on the segment $[t_0,t_0+b]$ there is exist a solution to the Cauchy problem (1), where $b=\min\{a,d\}$, $\int_{t_0}^{t_0+d} g(t)dt\le r$.

From the theory of differential inclusions is well-known [1,3] that if the above condition 2 $S_r(x_0)$ replaced by R^n, instead of condition 3 to assume measurability of a set-valued map $F(t,x)$ by $t\in T=[t_0,t_0+a]$, and instead of condition 4 to assume the condition is met

$\| F(t,x)\| \le g_1(t)\| x\| + g_2(t),, g_1(\cdot)\in L_1(T), g_2(\cdot)\in L_1(T),$ for all $x_0\in R^n$ the set of all solutions of the Cauchy problem (1) $H_T(x_0,F)$ is a non-empty compact set of $C^n(T)$.

In this paper, for a class of differential inclusions, we will study the properties of the set of solutions and some of their consequences concerning the structure of the reachability set and the of reachability sphere.

1. Statement of the problem. Research methods.

Let in (1) $F(t,x)=A(t)x+B(t)$, where $A:T\to R^{n\times n}$, $B:T\to P(R^n)$, i.e. consider the Cauchy problem

$$\dot{x}\in A(t)x+B(t), t\in T, x(t_0)=x_0. \qquad (2)$$

We denote the $H_T(x_0,A,B)$ – the set of all absolutely continuous solutions of problem (2). In the theory of differential inclusions, the set $X_T(\tau,x_0,A,B)=\{\eta\in R^n:\eta=x(\tau), x(\cdot)\in H_T(x_0,,A,B\}$ is of great interest, which is the reachability set of system (2) at time $\tau\in T$. Let $Z\in P(R^n)$. Denote by $X_T(\tau,Z,A,B)$ the reachability set of the system

$$\dot{x}\in A(t)x+B(t), t\in T, x(t_0)\in Z. \qquad (3)$$

It is clear that $X_T(\tau,Z,A,B)=\bigcup\limits_{\xi\in Z} X_T(\tau,\xi,A,B)$.

Let us put $\Sigma_T(\tau,Z,A,B)=\bigcup\limits_{s\le\tau, s\in T} X_T(s,\xi,A,B), \tau\in T$. The set $\sum_T(\tau,Z,A,B)$, is called the reachability sphere of system (3) in time $t=\tau-t_0$. Let us denote by $\Phi_A(t,\tau)$ the fundamental matrix of solutions of equation $\dot{x}=A(t)x, t\in T$. By definition

$$\frac{\partial\Phi_A(t,\tau)}{\partial t}=A(t)\Phi_A(t,\tau), t\in T, \Phi_A(\tau,\tau)=E,$$

where E is the unit nxn-matrix.

To study the properties of the solution set and the structure of the reachability sets and the reachability sphere, we will use the methods of the theory of multivalued maps and differential inclusions.

2. The main results.

From the results of the theory of differential inclusions it follows.

Theorem 1. Let the elements of the matrix $A(t)$ be measurable and $\| A(t)\|\le a(t), t\in T$, where $a(\cdot)\in L_1(T)$, and the multi-valued map $B:T\to\Omega(R^n)$ be measurable and $\| B(t)\|\le\varphi(t), t\in T$, где $\varphi(\cdot)\in L_1(T)$. Then:

a) $H_T(x_0,A,convB)\in C\Omega(R^n(T)), \forall x_0\in R^n$.

b) $\lambda_1 H_T(\xi_1,A,convB)+\lambda_2 H_T(\xi_2,A,convB)=H_T(\lambda_1\xi_1+\lambda_2\xi_2,A,convB), \forall\lambda_1\ge 0,\lambda_2\ge 0.$

c) if $Z\in\Omega(R^n)$, то $H_T(Z,A,convB)\in\Omega(C^n(T)), H_T(convZ,A,convB)\in C\Omega(C^n(T))$,

where $H_T(Z,A,convB)=\bigcup\limits_{\xi\in Z} H_T(\xi,A,convB).$

Using the results of the theory of multivalued analysis, it is possible to obtain representations of the reachability set and the reachability sphere of system (3) through the fundamental matrix $\Phi(t,\tau)$ and the multivalued map $B:T\to P(R^n)$.

Theorem 2. Let the matrix function $A:T\to R^{nxn}$ and the multi-valued map $B:T\to P(R^n)$ satisfy the conditions of Theorem 2. Then:

$$X_T(t,Z,A,B)=X_T(t,Z,A,convB)=\Phi_A(t,t_0)Z+\int_{t_0}^{t}\Phi_A(t,\tau)B(\tau)d\tau, t\in T, \qquad (4)$$

$$\sum_T(t,Z,A,B)=\sum_T(t,Z,A,convB)=\bigcup_{s\le t,s\in T}[\Phi_A(s,t_0)Z+\int_{t_0}^{s}\Phi_A(s,\tau)B(\tau)d\tau],\ t\in T.\ (5)$$

Corollary 1. Let $Z\in C\Omega(\grave{R}^n)$. Then, under the conditions of Theorem 2, the set $X_T(t,Z,A,B)$ is a convex compact of R^n for all $t\in T$, and the multi-valued map $t\to X_T(t,Z,A,B)$ is continuous on T. If, then, Z is a strictly convex compact, and sets $convB(t)$ are strictly convex compacts for almost all $t\in T$, then set $X_T(t,Z,A,B)$ is a strictly convex compact for all $t\in T$.

Corollary 2. Let $Z\in P(R^n), T=[t_0,t_1],\ t_0<\tau<t_1$. Then, under the conditions of Theorem 2, the equality $X_{[t_0,t_1]}(t_1,Z,A,B)=X_{[\tau,t_1]}(t_1,X_{[t_0,\tau]}(\tau,Z,A,B),A,B)$ is valid.

Theorem 3. Let $A(t)\equiv A, B(t)\equiv B,\ t\in T=[t_0,t_1],\ B\in P(R^n),\ Z\in P(R^n)$. Then , if $AZ\subset -B$, then the equality $X_T(t_1,Z,A,B)=\sum_T(t_1,Z,A,B)$ is valid .

Corollary 3. Let the conditions of Theorem 3 be satisfied, with $Z\in C\Omega(R^n)$. Then the sphere of reachability $\sum_T(t_1,Z,A,B)$ is a convex compact.

3. Conclusion.

In this paper, we study the properties of a set of absolutely continuous solutions to the Cauchy problem for one class of differential inclusions. The structure of the reachability set and the reachability sphere are investigated. The obtained formulas (4) and (5) allow us to clarify the various properties of the considered sets. In addition to the above properties, in particular, the following formulas can be obtained for the support function of the reachability set $X_T(t,Z,A,B)$ and the reachability sphere $\sum_T(\tau,Z,A,B)$:

$$C(X_T(t,Z,A,B),\psi)=C(\Phi_A(t,t_0)Z,\psi)+\int_{t_0}^{t}C(\Phi_A(t,\tau)B(\tau),\psi)d\tau .$$

$$C(\sum_T(t,Z,A,B),\psi)=\sup_{s\le t,s\in T}[C(\Phi_A(s,t_0)Z,\psi)+\int_{t_0}^{s}C(\Phi_A(s,\tau)B(\tau),\psi)d\tau].$$

The studied properties of the considered system are of interest for linear state control systems.

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